Econ712 - Handout 2 Sol

# 1 Moderately timed review

## 1.1 Modeling

- Environment: Agents; Preferences/Payoffs; Technology; Information
- Equilibrium

## 1.2 Class material

- Consumption - labour decision
- Planner's allocation vs competitive equilibrium allocation

# 2 Dynamics and Difference equations

- Macro deals with problems that are dynamic in nature
- In discrete time, equilibrium characterized by difference equations
- Solution/characterization: Linearization; Phase Diagrams; Numerical

## 2.1 Some bivariate linear difference equation

Consider the following system

$$\Omega_{t+1} = (1-\delta)\Omega_t + L_t^c$$
$$L_{t+1}^c = (1+\delta+\beta)L_t^c + \gamma\Omega_t - \theta_0$$

Here $\delta \in (0,1), \beta \in (0,1), \gamma \geq 0$. Illustrate the system dynamics using a phase diagram with $\Omega_t$ on the horizontal axis and $L_t^c$ on the vertical axis. Analyze the regions where $\Delta\Omega_t > 0, \Delta\Omega_t = 0, \Delta\Omega_t < 0$, and do the same for $L_t^c$. Is there a steady state $(\bar{\Omega}, \bar{L}^c)$ and saddle path?

- It is always important to note which are state variables and which are choice variables. While not given in the problem, let's assume $\Omega_t$ is the state and $L_t^c$ is the choice.
- To construct phase diagram for this system, we need to find the equations for two phaselines,

  $$\Delta\Omega_t = \Omega_{t+1} - \Omega_t = 0 \tag{1}$$
  $$\Delta L_t^c = L_{t+1}^c - L_t^c = 0 \tag{2}$$

  Above the $\Delta\Omega_t = 0$ line, $\Omega_t$ will increase and below the $\Delta\Omega_t = 0$ line it will decrease. The same logic works for the dynamics of $L_t^c$.

- Using equations (1) and (2), we can determine the direction of the vector field and the sense of the steady state.

$$\Delta\Omega_t \geq 0 \iff L_t^c \geq \delta\Omega_t$$
$$\Delta L_t^c \geq 0 \iff L_t^c \geq -\frac{\gamma}{\delta+\beta}\Omega_t + \frac{\theta_0}{\delta+\beta}$$

- To determine whether there is a saddle path, we need to consider the stability of the system. Quite a bit of algebra, let's assume it exists. (We know that since $\gamma > 0$, there exists a path that leads to the steady state. The question is whether that path is unique.)

Figure 2 shows the phase diagram for the system of $(\Omega_t, L_t^c)$ and a saddle path, assuming one exists

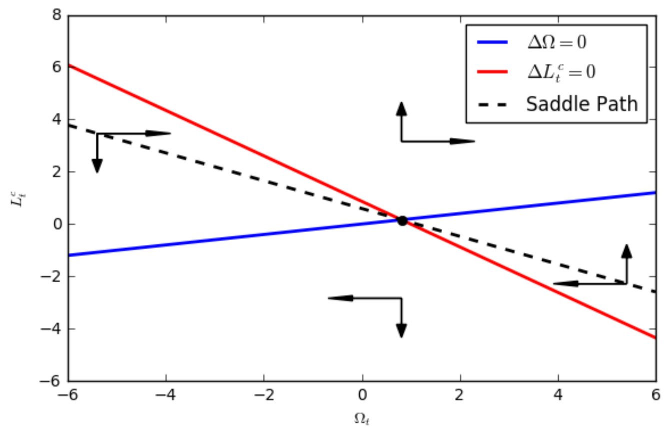


Figure 1: Phase Diagram $(\Omega_t, L_t^c)$

## 2.2 Finite consumption - saving

Consider an agent that lives for $T$ periods with time seperable utility. She ranks consumption each period according to $u(C)$, and discounts future consumption geometrically at rate $\beta$. Each period she is endowed with $w > 0$ units of consumption good. She has access to a perfect storage technology, whereby 1 unit of good saved today will give her 1 unit of good tomorrow. Assume $u(C)$ is strictly increasing, strictly concave, and continuously differentiable, with $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. In period 0, the agent manages to find treasures valued at $S$.

1. Derive the difference equations that characterize the agent's consumption and savings decisions.
   (a) Her problem: $\max_{\{c_t, s_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$ subject to $c_t + s_{t+1} = w + s_t$; $s_0 = S$; $c_t, s_{t+1} \geq 0$

(b) Clearly $c_t > 0$. Let $\mu_t$ be the multiplier on $s_{t+1} \geq 0$. FOCs:

$$\beta^t u'(c_t) = \lambda_t$$
$$\lambda_t - \mu_t = \lambda_{t+1}$$
$$\mu_t s_{t+1} \geq 0$$
$$c_t + s_{t+1} = w + s_t$$

This is a system of 4 difference equations and inequalities in 4 unknowns $c, s, \lambda, \mu$. Boundary conditions are $s_0 = S, \lambda_T = 0$

2. Let $T = 23$, $w = 1$, $u(c) = \log(c)$, $\beta = 0.99$, $S = 34$. Solve numerically for the optimal consumption and savings decisions.

   (a) Since $c_t << \infty\ \forall t$, $\lambda_{T-1} > 0$. $\lambda_T = 0$ then implies that $\mu_{T-1} > 0$ i.e. $s_T = 0$

   (b) Note that $\lambda_t \geq \lambda_{t+1} \iff u'(c_t) \geq \beta u'(c_{t+1})$, with equality if $s_{t+1} > 0$

   (c) We will solve a more general problem of finding the optimal consumption and savings policy $c_t^*(s), s_{t+1}^*(s)$ for all periods, given an arbitrary start of period savings:

      i. For period $T-1$, $s_T^*(s) = 0$ and $c_{T-1}^*(s) = w + s$

      ii. For period $T-2$, $s_{T-1}^*(s)$ solves $u'(w + s - s_{T-1}^*) \geq u'(c_{T-1}^*(s_{T-1}^*))$. That is, first solve the equality. If the resulting $s_{T-1}^* < 0$, set it to 0. Then $c_{T-2}^*(s) = w + s - s_{T-1}^*(s)$

      iii. Iterate backwards: $s_{t+1}^*(s)$ solves $u'(w + s - s_{t+1}^*) \geq u'(c_{t+1}^*(s_t^*))$ and $c_t^*(s) = w + s - s_{t+1}^*(s)$

      iv. Given the policy functions, for our specific starting point $s_0 = 34$, solve forwards. That is, $s_1 = s_1^*(34)$, $s_2 = s_2^*(s_1)$, and so on

   (d) A small note on the shooting method:

      i. If we had $u'(c_t) = \beta u'(c_{t+1})\, \forall t$, we could start with a guess of $s_{T-1}$, which then imply a $s_{T-2}$ by $u'(w + s_{T-2} - s_{T-1}) = u'(w + s_{T-1} - s_T)$. Continuing in this manner, we would obtain a sequence down to $s_0$. If $s_0 = S$, then our guess is correct and we have found the optimal sequence. If not, we change our guess of $s_{T-1}$

      ii. The presence of the inequality $u'(c_t) \geq \beta u'(c_{t+1})$ complicates this approach a bit

# 3 Setting up a model

State the Consumer Problem and define the Competitive Equilibrium for the following:

Consider an overlapping generations economy of 3-period-lived agents. Denote these periods as *young, mid, old.* At each date $t \geq 1$, a measure 1 of new young agents enter the economy, each endowed with $w_1$ units of the consumption good when young, $w_2$ units when mid, and $w_3$ units when old. The consumption good is non-storable. Consumption preference is described by $\ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$. At time $t = 1$, there is an unit measure of old agents, each endowed with $w_3$ units of the consumption good; and an unit measure of mid agents, each endowed with $w_2$ units of the consumption good at $t = 1$ and $w_3$ units at $t = 2$. Additionally, each inital old agent is endowed with 1 unit of fiat currency.

- Consumer Problem

  - Young agents can trade with mid agents. Hence we can include a 1-period bond $b$ with price $Q$, available only when young and mid.

- Initial old: $\max_{c_1^{-1}} \ln c_1^{-1}$ s.t. $c_1^{-1} \leq w_3 + 1/P_1$
- Initial mid: $\max_{c_1^0, c_2^0} \ln c_1^0 + \ln c_2^0$ s.t.
- $c_1^0 + M_2^0/P_1 + Q_1 b_2^0 \leq w_2$
- $c_2^0 \leq w_3 + b_2^0 + M_2^0/P_2$
- Other generations:
- $\max_{c_t^t, c_{t+1}^t, c_{t+2}^t, b_{t+1}^t, b_{t+2}^t, M_{t+1}^t, M_{t+2}^t} \ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$ s.t.
- $c_t^t + M_{t+1}^t/P_t + Q_t b_{t+1}^t \leq w_1$
- $c_{t+1}^t + M_{t+2}^t/P_{t+1} + Q_{t+1} b_{t+2}^t \leq M_{t+1}^t/P_{t+1} + b_{t+1}^t + w_2$
- $c_{t+2}^t \leq M_{t+2}^t/P_{t+2} + b_{t+2}^t + w_3$
- $(c_t^t, c_{t+1}^t) \geq \mathbf{0}$; $(M_{t+1}^t, M_{t+2}^t) \geq \mathbf{0}$; $w_1/Q_t \geq b_{t+1}^t \geq -(M_{t+1}^t/P_{t+1} + w_2)$; $(M_{t+1}^t/P_{t+1} + b_{t+1}^t + w_2)/Q_{t+1} \geq b_{t+2}^t \geq -(M_{t+2}^t/P_{t+2} + w_3)$

- Competitive Equilibrium
  - Allocation $\{c_t^{t-2}, c_t^{t-1}, c_t^t, M_{t+1}^t, M_{t+2}^t, b_{t+1}^t\}$ and prices $\{P_t, Q_t\}$ s.t. given prices, allocation solves consumer problem (agents optimize), and markets clear:
  - $c_t^t + c_t^{t-1} + c_t^{t-2} \leq w_1 + w_2 + w_3$
  - $M_{t+1}^t + M_{t+1}^{t-1} = 1$
  - $b_{t+1}^t + b_{t+1}^{t-1} = 0$