ECON 712A: Macroeconomic Theory - Section Handout 3

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Correction from Handout 1:

- Duong's office hours are Tuesday 10:30 AM - 11:45 AM in 6439 Social Science.
- Alex's office hours are Thursday at 2:15 PM - 3:30 PM in 6473 Social Science.

Content Review

- Labor/leisure notes:
  - The **Lucas critique** argues that relationships observed in historical data may not be invariant to changes in government policy. In lecture, we saw an example of how the Lucas critique applies to a linear regression of labor supply on productivity.
  - The **First Welfare Theorem** states that under appropriate assumptions, a competitive equilibrium is pareto optimal.
  - The **Second Welfare Theorem** states that under appropriate assumptions, any pareto optimal allocation can be achieved as a competitive equilibrium with appropriate lumpsum taxes and transfers.
- Kehoe notes:
  - Similar OG framework as introduction, but old agents have endowment $w_2 > 0$.
  - **Discount factor** $\beta \in [0, 1]$ in utility function: $U(c_t^t, c_{t+1}^t) = \ln(c_t^t) + \beta \ln(c_{t+1}^t)$
  - **Inside vs. outside money**: $M_{t+1}^t \in \mathbb{R}$
  - **Intertemporal budget constraint**: inside money allows us to consolidate budget constraints: $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t w_1 + p_{t+1} w_2$
  - **Excess demand** is demand minus endowment: $x_1(p_t, p_{t+1}) = c_t^t - w_1$ and $x_2(p_t, p_{t+1}) = c_t^{t+1} - w_2$.
  - **Market clearing** in excess demand space implies that $\frac{x_2(p_{t-1}, p_t)}{x_1(p_t, p_{t+1})} = -1$.
  - **Offer curve** is the locus of optimal excess demands as the price ratio varies between 0 and ∞: $\frac{x_2(p_t, p_{t+1})}{x_1(p_t, p_{t+1}} = -\frac{p_t}{p_{t+1}}$.
  - Walk through the top panel of Figure L3.2.

A couple things from problem set 1

- Where do **prices** show up?
  - The planner problem is to maximize present value of utility subject to resource constraint.
  - No prices in the planner problem.
  - Competitive equilibrium: Taking prices as given, households optimize subject to budget constraint and firms optimize subject to production technology. Apply market clearing to find prices.
  - No prices appear in market clearing conditions.
- In dynamic optimization problems, there are **control variables** and **state variables**.
  - Each period, agents wake up, look at their state variables (e.g., wealth today), make decision about their control variables (e.g., consumption today and wealth tomorrow), and then go back to sleep.
  - We'll talk a lot more about control and state variables in second quarter.
- **Discount factor**, **geometric discounting**, **interest rates**, and **present value**:
  - Discount factor $\beta \in (0,1)$ is a model primitive that captures impatience. Agents would prefer consuming now.
  - Geometrically discounting of steam of future consumption: $U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t)$
  - "Geometric" because of the geometric series: $\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$.
  - Why geometric discounting? Time consistent. Other ways to discount include hyperbolic discounting (time inconsistent; outside scope of this class).
  - Rate of return r vs. gross interest rates R: $r + 1 = R$.
  - Discount factor is a primitive (exogenous) and risk-free interest rate is a price (endogenous), but they are often related in deterministic representative agent models: $\beta = \frac{1}{R^f}$.
  - Using gross interest rate R, present value of payoff x in t periods: $PV(x) = R^{-t} x_t$.
  - Using gross interest rate R, present value of stream of payoffs $y = \{y_t\}_{t=0}^{T}$: $PV(y) = \sum_{t=0}^{T} R^{-t} y_t$.
- Once we start going through "Government Policy in an OG Environment" in lecture, take a look at the solution to problem 5.

Trade Offer Curves

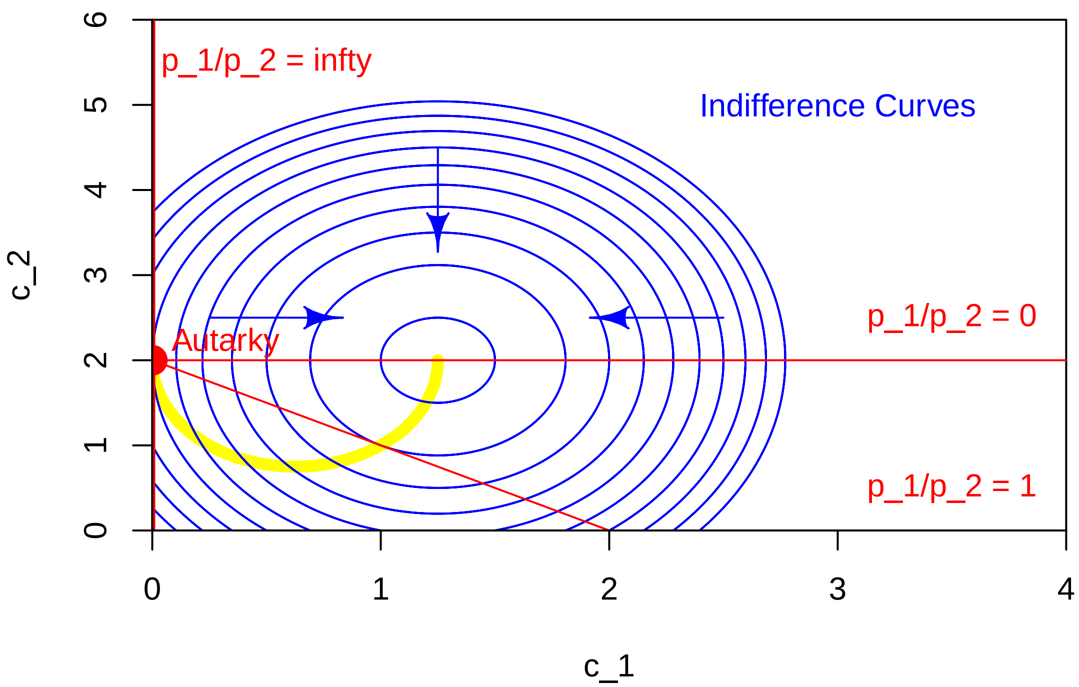
Plot the trade offer curves for the following utility functions where the endowment is (w_1, w_2) for goods 1 and 2, respectively.

(a) $U = 10c_1 - 4c_1^2 + 4c_2 - c_2^2, (w_1, w_2) = (0, 2)$

In the first figure, we can see that the utility function is an ellipsoid and thus the indifference curves are concentric ellipses centered at $(5/4, 2)$. Higher utility is associated with ellipses closer to $(5/4, 2)$. The budget constraint with a price ratio (p_1/p_2) of infinity is represented as a vertical line at $c_1 = 0$. The highest utility the agent can get is at the autarky point, which is tangent to the indifference curve associated with $\bar{u} = 4$. As the price ratio decreases, the slope of the budget constraint decreases. The agent will choose consumption at the point allow the budget constraint that is tangent to highest utility ellipse. For example, at $p_1/p_2 = 1$, the agent will choose to consume $(1, 1)$ which is tangent to the indifference ellipse associated with $\bar{u} = 9$. At $p_1/p_2 = 0$, the budget constraint is horizontal line at $c_2 = 2$. Thus, the agent will consume $(5/4, 2)$ achieving their maximum utility of $\bar{u} = 10.25$. Thus, the trade offer curve traces the lower half of an ellipse starting at the autarky point through the tangent points to the concentric ellipse and ending at the maximum utility point.

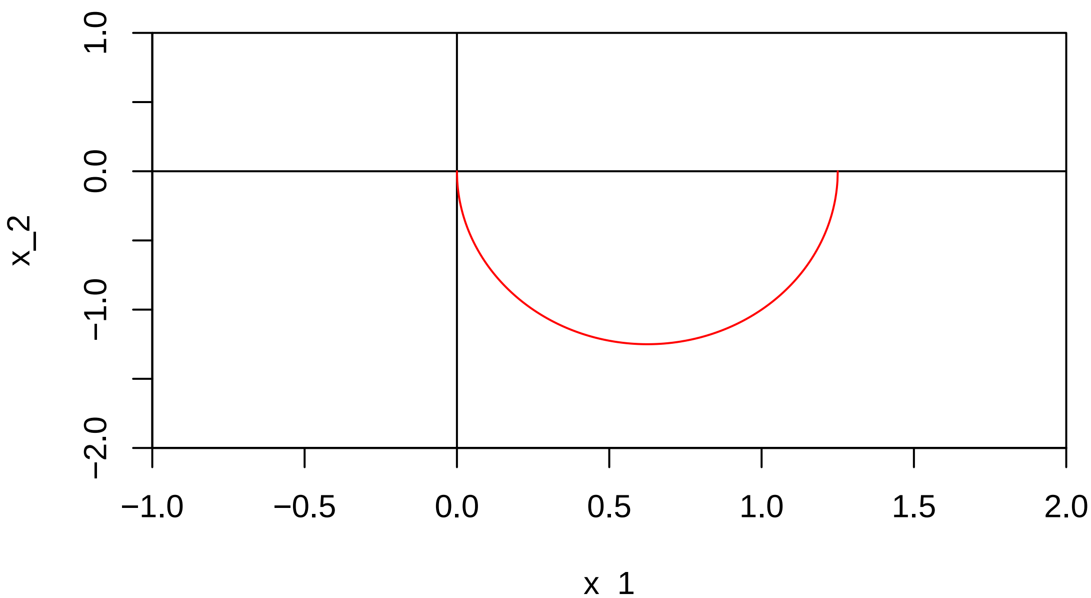
The second figure translates the trade offer curve from consumption space to excess demand space.

Consumption Space



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)

Excess Demand Space

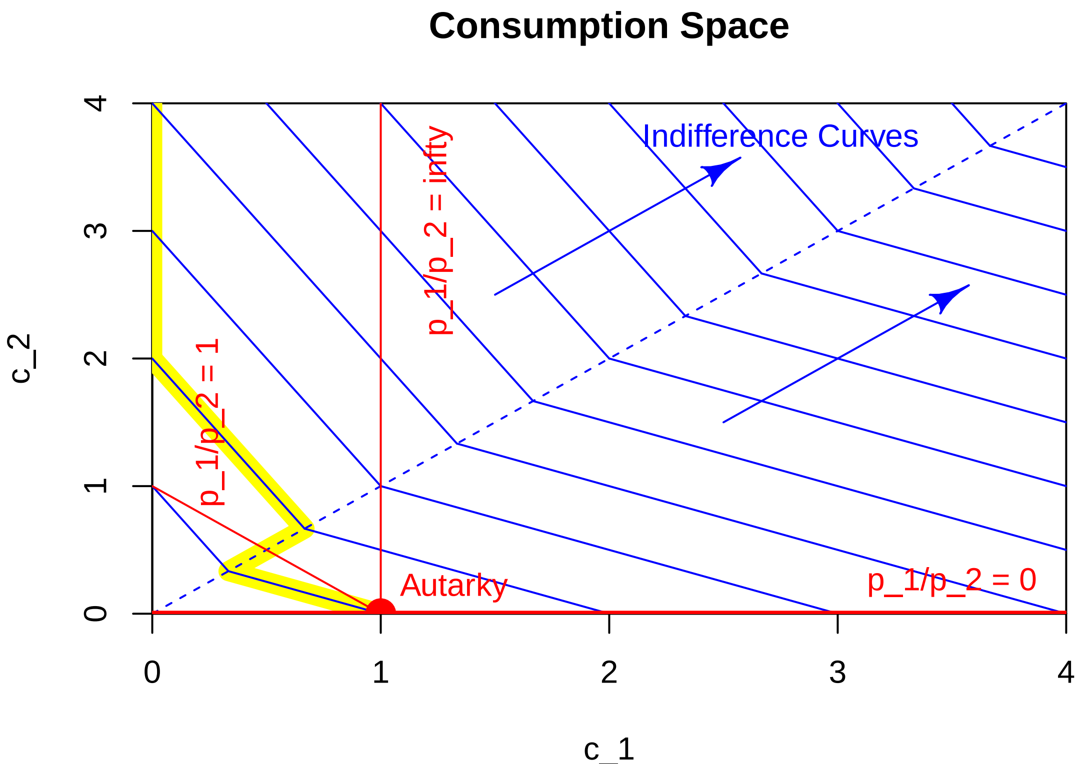


Trade offer curve (red)

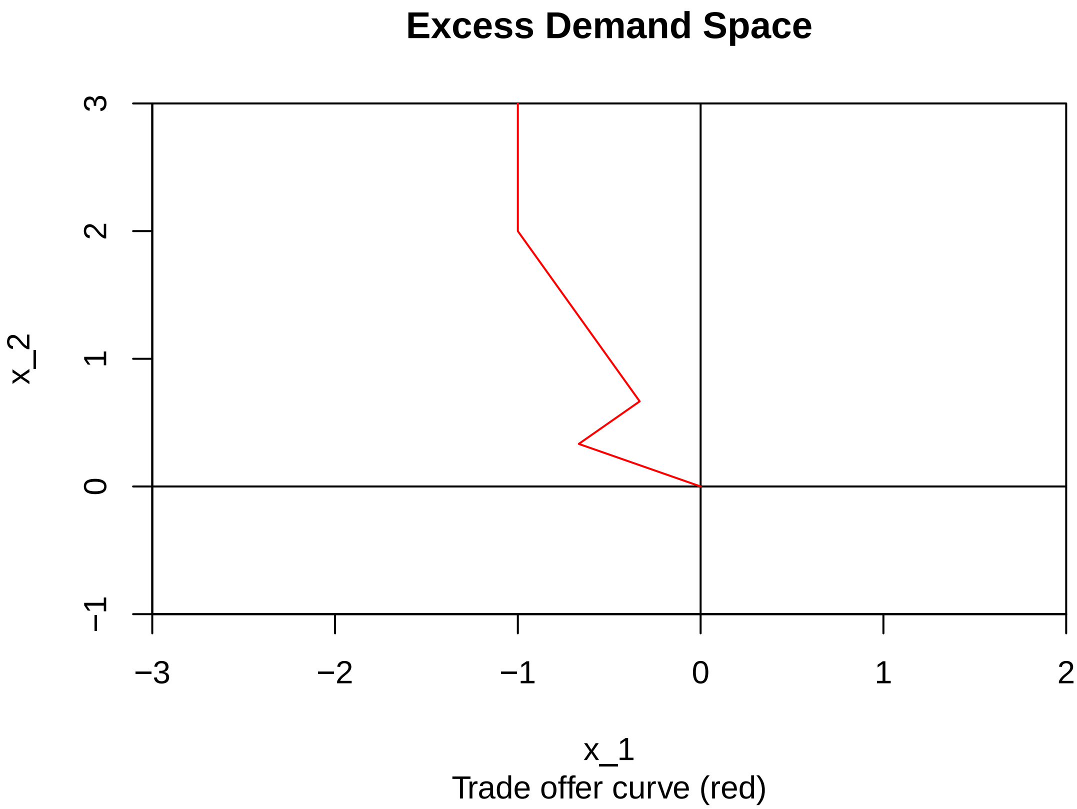
(b) $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 0)$

In the first figure, we see that the indifference curves are similar to the perfect complements case discussed in section. However, the lines below the identity line have a slope of 1/2 and the lines above the identity line have a slope of 2. At a price ratio (p_1/p_2) of zero, the budget constraint is a horizontal line at $c_2 = 0$. To get the highest indifference curve, the agent will consume infinity units of c_1 and zero units of c_2. Between price ratios of zero and 1/2, the agent will consume at the autarky point of $(1, 0)$. The agent will continue to consume the autarky point until the price ratio is 1/2, at which point the budget constraint will lay on top of the lower half of the indifference curve associated with $\bar{u} = 1/2$. Thus, the agent is indifferent between consuming at all points on the line segment between $(1, 0)$ and $(1/3, 1/3)$. At price ratios between 1/2 and 2, the agent will consume along the identity line. At a price ratio of 2, the budget constraint sits on top of the upper half of the indifference curve associated with $\bar{u} = 1$. At this price ratio, the agent is indifferent between consuming the points along the line segment between $(2/3, 2/3)$ and $(0, 2)$. At price ratio higher than 2, the agent consumes all c_2 at the highest amount possible. At a price ratio of infinity, the agent consumes infinity units of c_2.

The second figure translates the trade offer curve from consumption space to excess demand space.



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)

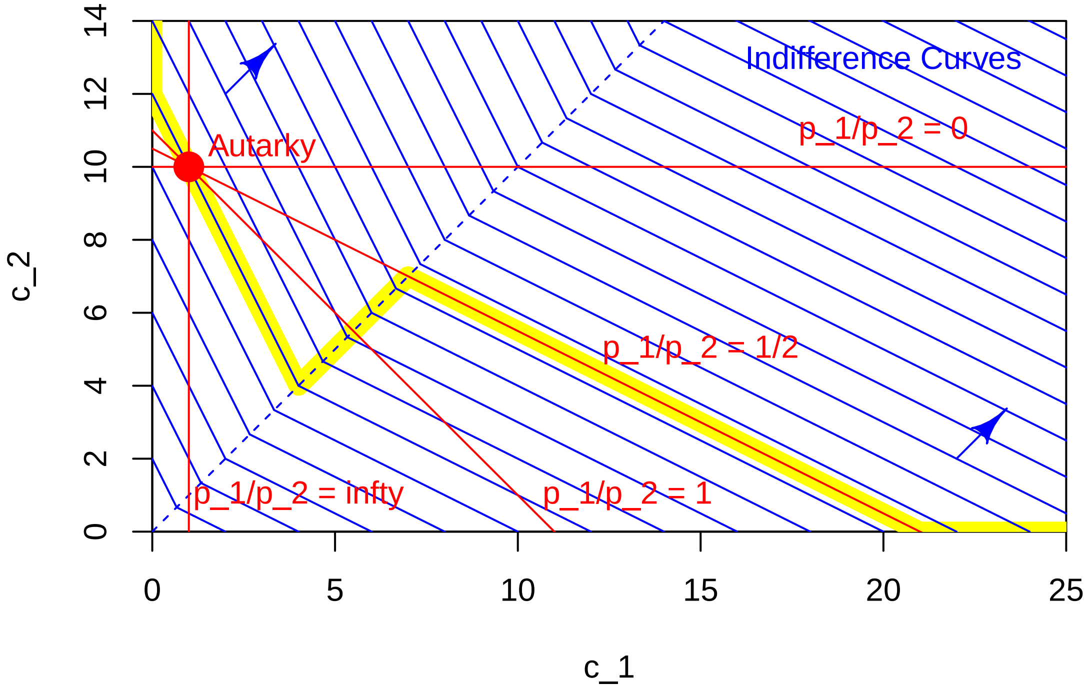


(c) $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 10)$

In the first figure, we see that the indifference curves are the same as in (b). At an price ratio (p_1/p_2) of 0, the agent consumes infinity units of c_1 and zero units of c_2. As the price ratio increases, the agent consumes less and less units of c_1. At a price ratio of 1/2, the budget constraint sit on top of the lower half of an indifference curve at $\bar{u} = 10.5$. The agent is indifferent between consuming any point between $(21, 0)$ and $(7, 7)$. For price ratios between 1/2 and 2, the agent consumes along the identity line. At a price ratio of 2, the budget constraint lays on top of the upper half of an indifference curve associated with $\bar{u} = 6$. The agent is indifferent between consuming at points along the line segment between $(4, 4)$ and $(12, 0)$. This segment passes through the autarky point. For higher price ratios, the agent consumes zero units of c_1 and more and more units of c_2. At a price ratio of infinity, the agent would consume infinity units of c_2 and zero units of c_1.

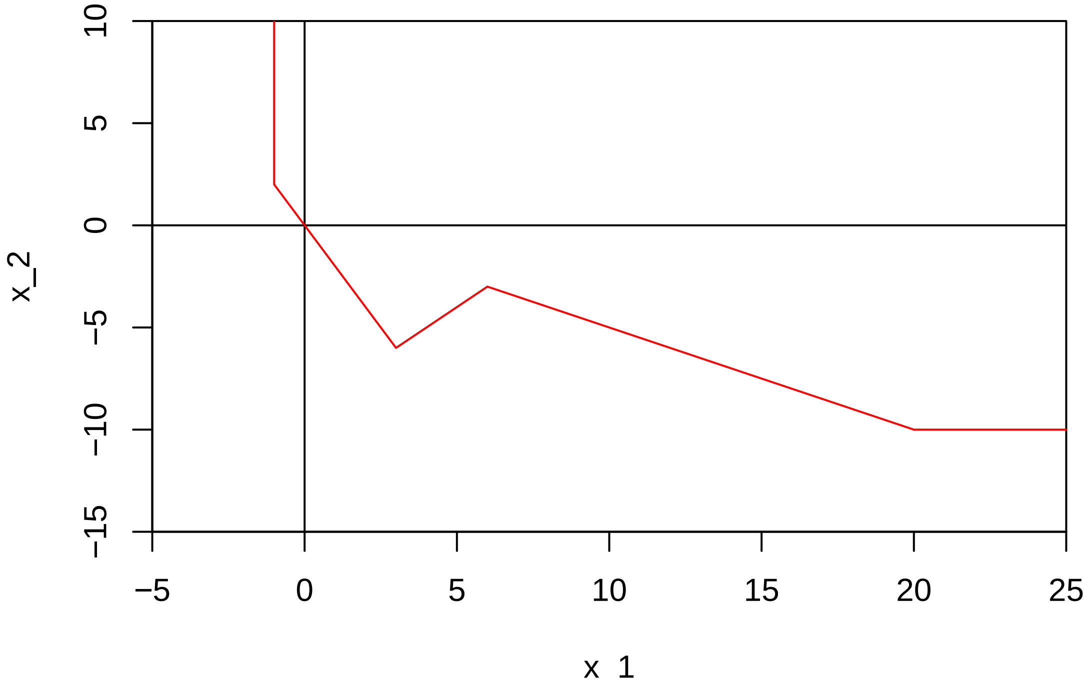
The second figure translates the trade offer curve from consumption space to excess demand space.

Consumption Space



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)

Excess Demand Space



Trade offer curve (red)

Gale's Pure Exchange Equilibrium of Dynamic Economic Models

Consider the following overlapping generations problem.[1] In each period t a new generation of 2 period lived households are born, and each generation has a unit measure. Each generation is endowed with $w_1 = 0$ in youth and $w_2 = 2$ in old age of nonstorable consumption goods. In each period the agents can buy or issue money $M_{t+1}^t \in \mathbb{R}$, to some outside agent. We assume that the market for money always clears. The utility function of a household of generation t is quadratic:[2]

$$U\left(c_t^t, c_{t+1}^t\right) = 10c_t^t - 4(c_t^t)^2 + 4c_{t+1}^t - (c_{t+1}^t)^2 \tag{1}$$

where $\left(c_t^t, c_{t+1}^t\right)$ is consumption of a household of generation t in youth (i.e. in period t) and old age (i.e in period $t+1$). We ignore the initial old.

1. Write down the problem of generation t with a consolidated budget constraint.

   The two period budget constraints are: $p_t c_t^t + M_{t+1}^t = p_t w_1$ and $p_{t+1} c_{t+1}^t = p_{t+1} w_2 + M_{t+1}^t$. Inserting for d_t, letting $q_t = \frac{p_t}{p_{t+1}}$ and inserting for $w_1 = 0, w_2 = 2$ we then get the following problem:

$$\max_{(c_t^t, c_{t+1}^t) \geq 0} 10c_t^t - 4(c_t^t)^2 + 4c_{t+1}^t - (c_{t+1}^t)^2 \tag{2}$$

$$s.t.\ q_t c_t^t + c_{t+1}^t = 2 \tag{3}$$

2. Find the Euler equation (intertemporal optimality condition) and eliminate prices from your expression.

   We get the following Lagrangian:

$$\mathcal{L}(c_t^t, c_{t+1}^t) = 10c_t^t - 4(c_t^t)^2 + 4c_{t+1}^t - (c_{t+1}^t)^2 - \lambda(q_t c_t^t + c_{t+1}^t - 2). \tag{4}$$

   Taking FOCs with respect to consumption and combining gives:

$$5 - 4c_t^t = q_t(2 - c_{t+1}^t)$$

   Next, solve for q_t in the budget constraint, insert into the Euler equation and rearrange:

$$q_t = -\frac{c_{t+1}^t - 2}{c_t^t} \tag{5}$$

$$\implies 5 - 4c_t^t = -\frac{c_{t+1}^t - 2}{c_t^t}(2 - c_{t+1}^t) \tag{6}$$

$$\iff (5 - 4c_t^t)c_t^t = (2 - c_{t+1}^t)^2 \tag{7}$$

3. Use the market clearing condition for goods to find the two steady state consumption allocations for the young and old in equilibrium.

   Market clearing in this economy is just

$$c_{t+1}^{t+1} + c_{t+1}^t = 2 \tag{8}$$

   Thus, inserting for consumption when old in the Euler equation we get:

$$(5 - 4c_t^t)c_t^t = (2 - 2 + c_{t+1}^{t+1})^2 \tag{9}$$

$$\implies (5 - 4c_t^t)c_t^t = (c_{t+1}^{t+1})^2 \tag{10}$$

[1] Gale, D. (1973). *Pure exchange equilibrium of dynamic economic models.* Journal of Economic Theory, 6(1), 12-36.

[2] Quadratic utility is still used occasionally, e.g. Cochrane, J. (2014) *A Mean-Variance Benchmark For Intertemporal Portfolio Theory*, Journal of Finance.

Table 1: 2 period cyclical equilibrium

	t	t+1	t+2
Consumption for young	c_1	$\hat{c}_1$	c_1
Consumption for old	c_2	$\hat{c}_2$	c_2

Next, by enforcing steady state ($c_t^t = \bar{c}_1, c_{t+1}^t = \bar{c}_2 \ \forall t$) we get:

$$5\bar{c}_1 - 4\bar{c}_1^2 = \bar{c}_1^2 \tag{11}$$

Of course, this equation has two solutions, $\bar{c}_y = 0$ or $\bar{c}_y = 1$. Note that one steady state gives autarky (as it always is in OG models), while the other steady state gives perfect consumption smoothing, as the planner solution gave in the previous problem set.

4. Find all 2-period cyclical*(!)* competitive equilibrium (e.g. $c_t^t = c_{t+2}^{t+2}, c_{t+1}^t = c_{t+3}^{t+2} \ \forall t$).

   A 2-period cyclical equilibrium is where the young in period t has the same consumption as the young in $t+2$. Thus, there are four free variables (see table 1), so the question is how we pin those down. From equation 10 we have incorporated optimality and market clearing. We insert for the implications of cyclicality to get:

$$(5 - 4c_1)c_1 = (\hat{c}_1)^2 \ \forall s = t, t+2, t+4, \dots \tag{12}$$

   But of course, equation 10 must hold in 'odd' periods as well:

$$(5 - 4\hat{c}_1)\hat{c}_1 = (c_1)^2 \ \forall s = t+1, t+3, t+5, \dots \tag{13}$$

   These two equations define two ellipses in the $c_1, \hat{c}_1$ space. The solutions two these two equations that satisfy market clearing describe all 2-period cyclical equilibria.

$$(c_1, \hat{c}_1) = \left\{ (\frac{5-\sqrt{5}}{6}, \frac{5+\sqrt{5}}{6}), (\frac{5-\sqrt{5}}{6}, \frac{5+\sqrt{5}}{6}), (0,0), (1,1) \right\} \tag{14}$$

$$(c_1, \hat{c}_1) = \{(0.46, 1.21), (1.21, 0.46), (0,0), (1,1)\} \tag{15}$$

   By enforcing market clearing ($c_2 = 2 - c_1, \hat{c}_2 = 2 - \hat{c}_1$) we then get:

$$(c_2, \hat{c}_2) = \{(1.54, 0.79), (0.79, 1.54), (2,2), (1,1)\} \tag{16}$$

   And of course, the last two equilibria only describe the two stationary cyclical equilbria. The first two describes two competitive equilibria with *endogenous business cycles*. Note that we have now demonstrated that a competitive monetary equilibrium without uncertainty and rational expectations can have business cycles.

5. In the excess demand space, draw the offer curve and market clearing. Show that, depending on the initial price, we can get an equilibrium where excess demand 'jump' around the steady state, instead of smoothly converging.

   The indifference curve is given by:

$$\{c_t^t, c_{t+1}^t\} \in \mathbb{R} : 10c_t^t - 4(c_t^t)^2 + 4c_{t+1}^t - (c_{t+1}^t)^2 = \bar{U}, \tag{17}$$

   and note that this defines an ellipsis. The MRS is given by:

$$\frac{U_1}{U_2} = MRS = \frac{5 - 4c_t^t}{2 - c_{t+1}^t}, \tag{18}$$

   and we see that the indifference curves (ellipses) centers around $c_t^t = 1.25, c_{t+1}^t = 2$.

It's fairly straight forward to find the offer curves graphically, by changing the price ratio q_t (rotating the budget constraint around the endowment). Market clearing, rewritten for excess demand gives: $x_1(q_t)+x_2(q_{t+1})=0$ Figure 1 shows the offer curves and market clearing, as well as how the indifference curves are obtained

If excess demand for the old initially is $\tilde{x}_2$ we move into the two period cyclical equilibrium. If the excess demand in a period is slightly higher, $\hat{x}_2$, we converge into the non-autarkic steady state.

Note, from equation 7 we can find the offer curve explicitly: $x_1 = c_t^t, x_2 = c_{t+1}^t - 2$ which gives:

$$x_1(5-4x_1) = \pm(x_2)^2 \tag{19}$$

So for $x_2 = 0$ we have $x_1 = \{0, 5/4\}$. Drawing this for the 'minus part' we get the same as we did doing the graphical analysis (figure 1)

Figure 1: Equilibrium in the benchmark model