Econ712 - Handout 4 Sol

1 Competitive equilibrium with social security

1.1 Environment

- Demographics: Discrete time, 2 period lived households, population growth n
- Technology:
  - Endowments (w_1, w_2) of non-storable consumption goods in youth and old age
  - Commitment technology that allows trade to take place across generations
- Prefences:
  - Initial old: $u(c_1^0)$
  - Generation t: $u(c_t^t) + \beta u(c_{t+1}^t)$
  - Assume $\lim_{c\to 0} u'(c) = \infty$
  - Assume $\beta u'(w_2) > u'(w_1)$
- Social security system: the young pay lumpsum taxes $\tau \in [0, w_1)$ and recieve transfers b when old

1.2 Equilibrium

1. Set up the household's optimization problem

   The problem for the inital old is trivial. For generation t:

$$\max_{c_t^t, c_{t+1}^t} \left\{u(c_t^t) + \beta u(c_{t+1}^t)\right\} \quad s.t.$$

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t(w_1 - \tau) + p_{t+1}(w_2 + b)$$

2. What is the government's budget constraint?

$$bp_t = \tau p_t(1+n)$$

3. Define a Competitive Equilibrium with Social Security

   A CE with SS is a sequence of allocations $\{c_1^0, \{c_t^t, c_{t+1}^t\}\}$, prices $\{p_t\}$, and policies $\{\tau, b\}$ such that

   (a) Given prices and policies, the allocations are optimal for the households

   (b) Markets clear for all periods, i.e.

$$(1+n)c_t^t + c_t^{t-1} = (1+n)w_1 + w_2 \quad \forall t$$

   (c) The gov budget constraint holds for all periods

1.3 Characterization

1. Argue that autarky is the only equilibrium

   Fix τ and b. FOCs of the household problem imply that

   $$u'(c_t^t) = \beta u'(c_{t+1}^t)\frac{p_t}{p_{t+1}}$$

   Along with the budget constraint $p_t c_t^t + p_{t+1}c_{t+1}^t = p_t(w_1 - \tau) + p_{t+1}(w_2 + b)$, this gives the excess demand curves $x_t^t(\frac{p_t}{p_{t+1}})$ and $x_{t-1}^t(\frac{p_t}{p_{t+1}})$. Note that we can normalize $p_1 = 1$.

   Now for the initial old, they will consume their endowment, i.e. $x_1^0(\frac{1}{p_2}) = 0$. Market clearing requires $x_1^1(\frac{1}{p_2}) = 0$, which pins down p_2. From the budget constraint, $c_2^1 = w_2 + b$ since $c_1^1 - (w_1 - \tau) = x_1^1 = 0$, i.e. $x_2^1(\frac{1}{p_2}) = 0$. Continuing in this fashion, we will have that all agent will consume only their endowments net of transfers in equilibrium.

2. Denote the lifetime utility under autarky by

   $$V(\tau) = u(w_1 - \tau) + \beta u(w_2 + \tau(1+n))$$

   Starting from no social security system ($\tau = 0$), can a marginal increase in τ increase lifetime utility? Does the introduction of the social security system pareto improve over (strict) autarky?

   Our object of interest is

   $$\begin{aligned}\frac{dV(\tau)}{d\tau}\ |_{\tau=0} &= \{-u'(w_1 - \tau) + \beta u'(w_2 + \tau(1+n))(1+n)\}\ |_{\tau=0} \\ &= -u'(w_1) + \beta u'(w_2)(1+n)\end{aligned}$$

   This object is strictly positive $\iff$ $1 + n > u'(w_1)/\beta u'(w_2)$ which holds by our assumption. So a marginal increase in τ from $\tau = 0$ (strictly) increases lifetime utility, for all agents born from period 1. Moreover, the initial old is gaining transfers, so they are also better off. So the introduction of social security pareto improves over autarky.

3. What is the optimal social security tax and transfers?

   Define the optimal as one which maximizes lifetime utility. FOCs give

   $$-u'(w_1 - \tau^*) + \beta u'(w_2 + \tau^*(1+n))(1+n) = 0$$

2 Computing Life Cycle model

Consider a more general life cycle model with J period-lived agents. Newly born agents are endowed with no capital, but can subsequently save in capital which they can rent to firms at rate r. A worker of age j supplies labour $l_j \in [0,1]$ and pays proportional security tax on her labour income $\tau w e_j l_j$, where e_j is an age efficiency profile. For $j \geq J^R$, the worker retires ($l_j = 0$) and receives pension benefits b.

Preferences are given by

$$\sum_{j=1}^{J}\beta^{j-1}u(c_j, l_j) = \sum_{j=1}^{J}\beta^{j-1}\left(\frac{c_j^{\gamma}(1-l_j)^{1-\gamma}}{1-\sigma}\right)^{1-\sigma}$$

We will go through the steps to compute a stationary equilibrium of this economy.

2.1 Firm block

The production tech is $Y = F(K, L) = K^\alpha L^{1-\alpha}$. Capital depreciates at δ. Labor and capital markets are perfectly competitive, so that $w = F_2(K, L)$ and $r = F_1(K, L) - \delta$

2.2 Household block

The problem of the household:

$$\max \sum_{j=1}^{J} \beta^{j-1} \left(\frac{c_j^\gamma (1-l_j)^{1-\gamma}}{1-\sigma} \right)^{1-\sigma} \qquad s.t.$$

$$\begin{aligned} c_j + k_{j+1} &= (1-\tau) w e_j l_j + (1+r) k_j \quad j = 1, \ldots, J^{R-1} \\ c_j + k_{j+1} &= b + (1+r) k_j \quad j = J^R, \ldots, J \end{aligned}$$

First note that the optimal l_j (if unconstrained) is given by

$$l_j = \frac{\gamma(1-\tau)e_j w - (1-\gamma)\left[(1+r)k_j - k_{j+1}\right]}{(1-\tau)e_j w}$$

We could solve this problem in a variety of ways:

1. Euler Equation approach:
   (a) With $k_{j+1} \geq 0$, the consumption Euler inequality is

   $$u_1(c_j, l_j) \geq \beta u_1(c_{j+1}, l_{j+1}) [1+r]$$

   (b) With the boundary conditions $k_1 = k_{J+1} = 0$, we can solve this by solving for the policy functions $c_j(k_j)$ and $k_{j+1}(k_j)$ (recall handout 2)
2. Value Function approach:
   (a) Denote

   $$V_j(k) = \max_{c,l,k'} \left(\frac{c^\gamma (1-l)^{1-\gamma}}{1-\sigma} \right)^{1-\sigma} + \beta V_{j+1}(k') \quad s.t.$$

   $$\begin{cases} c + k' = b + (1+r)k & j = J^R, \ldots, J \\ c + k' = (1-\tau) w e_j l + (1+r)k & j = 1, \ldots, J^{R-1} \end{cases}$$

   Then $V_1(0)$ is the optimal lifetime utility of the agent
   (b) Let $V_{J+1}(k) = 0$. Solve numerically for V_J, then V_{J-1}, and so on
   (c) A byproduct of this are the policy functions $c_j(k_j)$ and $k_{j+1}(k_j)$

2.3 Aggregation

Aggregate K supply and L supply can be found be summing up capital and labour supply across generations, with appropriate generation weights. With growth rate n, the relative size of each cohort is given by $\psi_{i+1} = (1+n)^{-1}\psi_i$ with $\psi_1 = \tilde{\psi}$. Then

$$K = \sum_{j=1}^{J} \psi_j k_j$$

$$L = \sum_{j=1}^{J^R-1} \psi_j e_j l_j$$

2.4 Gov block

The gov budget constraint is

$$b = \frac{\tau w L}{\sum_{j=J^R}^{J} \psi_j}$$

2.5 Equilibrium and Algorithm

Finding a stationary equilibrium entails finding allocations $\{c_j, l_j, k_{j+1}\}$, prices (r, w), and policies (τ, b) such that

1. Household optimization is satisfied, given prices and policies
2. Firm's optimization satisfied
3. Markets clear: Capital supply = Capital demand; Labour supply = Labour demand; Goods supply = Goods demand
4. Gov BC is satisfied

The algorithm is as follows: Given some τ:

1. Outer block - Searching for prices: Guess w, r. Calculate the implied L and K supply, and b
   (a) Inner block: Solve for household optimal allocations as described above
   (b) Calculate aggregate L and K demand
2. If supply = demand, stop. Else update guess of w, r

Alternatively, one could guess L and K supply, which gives an implied w, r.