Econ712 - Handout 6

1 Diamond OG - Overview

- OG with capital accumulation
- Similar steps to solve as previous OG:
  - Planner problem: FOCs lead to system of difference equation - capital is dynamic
  - Competitive equilibrium:
    - Set up agents (household + firm) problems. Derive agents policies (optimization/FOCs)
    - Solve for prices to clear markets under agents' policies
    - Equilibrium allocations entail system of difference equation - capital is dynamic

Things to think about: Have we seen this environment somewhere prior to this week? Does the introduction of money in this environment matter?

2 Variation on the baseline Diamond OG

Take the environment as in class, with some changes in **bold**:

- Pop: Two period lived agents, **population growth** $N_{t+1} = (1+n)N_t$**,** with $N_0 = 1$
- Endowments: Agents have 1 unit of labour time when young
- Tech:
  - Production $Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$
  - Capital accumulation: $K_{t+1} = I_t + (1-\delta)K_t$. **Allow for general** $\delta \in [0,1]$
  - Consumption good and capital good can be exchanged 1 for 1
  - $\bar{K}_1$ given for initial old
- Pref: $\ln c_t^t + \beta \ln c_{t+1}^t$

1. Setup the planner's problem and derive the system of difference equation that characterize the planner's solution, given that the planner weighs each agent born in time t by $\gamma^t/(1+n)^t$. What is the steady state capital stock under the planner solution?

   (a) Planner:

$$\max \ln c_1^0 + \sum_{t=1}^{\infty} \gamma^t \left[\ln c_t^t + \beta \ln c_{t+1}^t\right] \quad s.t$$

$$K_{t+1} + N_{t-1}c_t^{t-1} + N_t c_t^t = K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t$$

   Let $k_t = K_t/N_t$. Dividing the time t resource constraint by N_t gives

$$k_{t+1}(1+n) + (1+n)^{-1}c_t^{t-1} + c_t^t = k_t^{\alpha} + (1-\delta)$$

FOC:

$$\gamma^t \frac{1}{c_t^t} = \lambda_t \tag{1}$$

$$\gamma^{t-1}\beta \frac{1}{c_t^{t-1}} = \lambda_t(1+n)^{-1} \tag{2}$$

$$\lambda_t(1+n) = \lambda_{t+1}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta) \tag{3}$$

(1) and (2) implies

$$\beta \frac{1}{c_t^{t-1}} = (1+n)^{-1}\gamma \frac{1}{c_t^t} \tag{4}$$

(1) and (3) implies

$$\frac{(1+n)}{c_t^t} = \gamma \frac{1}{c_{t+1}^{t+1}}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta) \tag{5}$$

(4), (5), and the resource constraint gives 3 difference equations in 3 unknowns. For the steady state, using (5):

$$(1+n) = \gamma(\alpha k^{\alpha-1} + 1 - \delta)$$
$$k = \left[\frac{\alpha}{(1+n)\gamma^{-1} - 1 + \delta}\right]^{\frac{1}{1-\alpha}}$$

2. Consider the decentralized competitive equilibrium of the above environment: Competitive firms rent capital from the old at r_t and hire labour from the young at w_t to maximize profits. Profits are then transfered to the old.

   (a) What are the conditions that characterize firm demand for capital and labour? Are there any profits?

      i. Assume that firms compensate households for capital depreciation. Firms $\max K_t^\alpha L_t^{1-\alpha} - w_t L_t - r_t K_t - \delta K_t$. FOCs give

$$r_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta$$
$$w_t = (1-\alpha) K_t^\alpha L_t^{-\alpha}$$

      ii. Plugging in for r_t and w_t gives that profits are zero

   (b) Setup the household problem and derive the household policies

      i. Ignoring profits, households solve

$$\max \ln c_t^t + \beta \ln c_{t+1}^t \quad s.t.$$

$$k_{t+1} + c_t^t = w_t$$
$$c_{t+1}^t = (r_{t+1} + 1)k_{t+1}$$

Consolidating the budget constraint gives

$$c_t^t + \frac{c_{t+1}^t}{1-\delta+r_{t+1}} = w_t$$

Utilizing log preferences, we get that

$$c_t^t = \frac{1}{1+\beta}w_t; c_{t+1}^t = \frac{\beta}{1+\beta}\frac{w_t}{1+r_{t+1}}$$

Then

$$k_{t+1} = \frac{\beta}{1+\beta}w_t$$

(c) Characterize the competitive equilibrium

i. The aggregate labour and capital stock are

$$\hat{L}_t = N_t$$
$$\hat{K}_{t+1} = N_t k_{t+1} \tag{6}$$

ii. Market clearing requires that the aggregate labour and capital be same as firm's demand. Plugging in firms and household policies into (6):

$$\hat{K}_{t+1} = N_t \frac{\beta}{1+\beta}(1-\alpha)\hat{K}_t^{\alpha} N_t^{-\alpha}$$

Define $\hat{k}_t = \hat{K}_t/N_t$. Rewrite the above equation as

$$\hat{k}_{t+1}(1+n) = \frac{\beta}{1+\beta}(1-\alpha)\hat{k}_t^{\alpha} \tag{7}$$

The competitive equilibrium is characterized by (7), from which we can back out prices and consumption allocations.

(d) Is the stationary capital stock in the competitive equilibrium efficient?

i. Using (7), the stationary stock is

$$\hat{k} = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

ii. Comparing this to the planner's optimal stationary capital, the two are equal iff

$$\frac{\beta(1-\alpha)}{(1+\beta)(1+n)} = \frac{\alpha}{(1+n)\gamma^{-1}-1+\delta}$$

If we can find a $\gamma > 0$ that satisfies the above equality, then the competitive equilibrium stationary capital stock is efficient.

$$(1+n)\gamma^{-1} - 1 + \delta = \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)}$$

$$(1+n)\gamma^{-1} = \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} + 1 - \delta$$

$$\gamma = (1+n)\left[\frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} + 1 - \delta\right]^{-1} > 0$$

given that $\delta \leq 1$