ECON 712A: Handout 7 - Solution

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Administrative Information

- Please fill out early TA evaluations (close on October 23rd).
- Midterm at 7:15 PM on November 1st. Bring pens/pencils. No need for a calculator. We will provide blue books for answers and scratch paper.

Content Review - Idiosyncratic Uncertainty

- **"Law of Large Numbers"**
  - Two-period OG model with unit measure of each generation.
  - Preferences: $U(c_t^t, c_{t+1}^t) = u(c_t^t) + u(c_{t+1}^t)$.s
  - Olds are unemployed (observable) with idiosyncratic probability π.
  - By "LLN", the mass of unemployed agents is also π.
  - Planners objective: $u(c_t^t) + \pi u(c_{t+1}^{u,t}) + (1-\pi)u(c_{t+1}^{e,t})$
  - Here, π is the mass of unemployed old agents and $1-\pi$ is the mass of employed old agents.
  - Ex-ante HH objective:

$$\pi[u(c_t^t) + u(c_{t+1}^{u,t})] + (1-\pi)[u(c_t^t) + u(c_{t+1}^{e,t})] = u(c_t^t) + \pi u(c_{t+1}^{u,t}) + (1-\pi)u(c_{t+1}^{e,t})$$

  - Here, π is probability of being unemployed and $1-\pi$ is probability of being employed.
- **Arrow-Debreu securities** are one-period assets that pay \$1 in a single future state.
- A **complete market** has assets that span the state space i.e. the number of assets with distinct payoffs is equal or greater than the number of states.
  - A market with an Arrow-Debreu security for each future state is complete.
  - Complete markets result in risk-sharing, i.e. consumption allocations depend on aggregate consumption not realization of idiosyncratic risk.

Next Steps - Incentive Compatibility

- What if the planner cannot observe idiosyncratic risk realizations?
- We solve the planner problem subject to incentive compatibility constraints.
- Incentive compatibility constraints results in allocations such that telling the truth gives higher utility than lying.

Private Commitment with Idiosyncratic Risk[1]

Consider the possibility of private commitment problems in a two period economy with a large number of agents who are subject to idiosyncratic income shocks. We will consider several different assumptions about the commitment technology.

Environment

- Population: There is a unit measure of type $i = B$ agents and a unit measure of type $i = S$ agents.
- Technology:
  - Type S agents have w_1 units of the consumption good at time $t = 1$ and 0 at $t = 2$.
  - Type B agents have 0 units of the consumption good at time $t = 1$. At time t=2, they have w_2^H in state $\theta = H$, which occurs with probability p, and w_2^L in state $\theta = L$, which occurs with probability $1 - p$.
- Preferences:
  - Let $c_t^i(\theta)$ denote consumption of a type i agent in period t in state θ
  - Type S agents have log utility $u(c_t^S(\theta)) = \ln(c_t^S(\theta))$ for both periods.
  - Type B agents have log utility $u(c_t^B(\theta)) = \ln(c_t^B(\theta))$ in period 1, but linear utility $u(c_t^B(\theta)) = c_t^B(\theta)$ in period 2.
  - Note: the state θ only matters in period 2.

Case 1 - Planner's problem with commitment

Assume agents can commit to all trades. State and solve the planner's problem given that she gives equal weight to every agent in the economy in the following steps:

1. State the planner's problem.

   Answer: Given the law of large numbers, the fraction of agents in state $\theta = H$ is p and in state $\theta = L$ is $1 - p$.

$$\max_{c_1^S, c_2^S, c_1^B, c_2^B(H), c_2^B(L)} \ln(c_1^S) + \ln(c_2^S) + \ln(c_1^B) + p(c_2^B(H)) + (1-p)(c_2^B(L))$$

   s.t.

$$\begin{aligned} c_1^S + c_1^B &= w_1 \\ c_2^S + pc_2^B(H) + (1-p)c_2^B(L) &= pw_2^H + (1-p)w_2^L \end{aligned}$$

[1] Based on the midterm from 2019.

2. What is the planner's allocation? Hint: with linear preferences there may be many allocations which satisfy the planner's solution.

   Answer:

   First suppose the parameters w_2^H and w_2^H given are such that $pw_2^H + (1-p)w_2^L \geq 1$. Then,

   Let λ be the Lagrange multiplier to the first period RC and μ be the Lagrange multiplier for the 2nd period RC. Then the social planner maximizes

$$\begin{aligned}\mathcal{L} = &\ln(c_1^S) + \ln(c_2^S) + \ln(c_1^B) + p(c_2^B(H)) + (1-p)(c_2^B(L)) \\ &+ \lambda(w_1 - c_1^S + c_1^B) \\ &+ \mu(pw_2^H + (1-p)w_2^L - c_2^S - pc_2^B(H) - (1-p)c_2^B(L))\end{aligned}$$

   Then, the FOCs are,

$$\begin{aligned}[c_1^S] &: \frac{1}{c_1^S} = \lambda \\ [c_1^B] &: \frac{1}{c_1^B} = \lambda \\ [c_2^S] &: \frac{1}{c_2^S} = \mu \\ [c_2^B(H)] &: p = p\mu \\ [c_2^B(L)] &: (1-p) = (1-p)\mu\end{aligned}$$

   Therefore,

$$\begin{aligned}\mu &= 1 \\ c_2^S &= 1\end{aligned}$$

   By the period 1 resource constraint,

$$c_1^B = c_1^S = \frac{w}{2}$$

   By the period 2 resource constraint, the optimal choices $c_2^B(H)$ and $c_2^B(L)$ are any combination that satisfy,

$$1 + pc_2^B(H) + (1-p)c_2^B(L) = pw_2^H + (1-p)w_2^L$$

   If $pw_2^H + (1-p)w_2^L < 1$, then, this SP solution violates non-negativity of consumption. Therefore,

$$c_2^S = pw_2^H + (1-p)w_2^L < 1$$

   and

$$c_2^B(H) = c_2^B(L) = 0$$

Case 2 - Bonds with commitment

Now consider a competitive equilibrium with a **non-state-contingent** private bond market. A person of type i can borrow or save in period $t = 1$ in a noncontingent bond a^i_{t+1} at price q. Specifically, if an agent of type i chooses $a^i_2 > 0$, then she gives up qa^i_2 goods at $t = 1$ and receives a^i_2 goods at $t = 2$ and if she chooses $a^i_2 < 0$, then she receives qa^i_2 goods at $t = 1$ and must pay back a^i_2 goods at $t = 2$. Note that neither prices nor assets depend on H or L.

3. Which type agent is a natural candidate for a borrower versus a saver and why? What are the optimization problems of each type agent?

   Answer:

   Given agent S has no endowment in period 2 and agent B has no endowment in period 1, they are naturally the savers and borrowers, respectively.

$$\max_{c_1^S, c_2^S} \ln(c_1^S) + \ln(c_2^S)$$
$$\text{s.t.}$$
$$c_1^S + qa_2^S = w_1$$
$$c_2^S = a_2^S$$

   and

$$\max_{c_1^B, c_2^B(H), c_2^B(L)} \ln(c_1^B) + p(c_2^B(H)) + (1-p)(c_2^B(L))$$
$$\text{s.t.}$$
$$c_1^B + qa_2^B = 0$$
$$c_2^B(H) = w_2^H + a_2^B$$
$$c_2^B(L) = w_2^L + a_2^B$$

4. What are the asset and goods market clearing conditions? Define a competitive equilibrium in this environment.

   Answer:

$$\begin{aligned} a_2^B + a_2^S &= 0 \\ c_1^S + c_1^B &= w_1 \\ c_2^S + pc_2^B(H) + (1-p)c_2^B(L) &= pw_2^H + (1-p)w_2^L \end{aligned}$$

   A competitive equilibrium in this environment is an allocation

$$\{c_1^S, c_2^S, c_1^B, c_2^B(H), c_2^B(L)\}, \{a_2^S, a_2^B\}$$

and price

$$\{q\}$$

such that:

(i) Savers and borrowers both optimally choose consumption and savings
(ii) Goods market and asset markets clear

5. Solve for a competitive equilibrium.

Answer:

Saver (type S):

$$\max \ln(w_1 - qa_2^S) + \ln(a_2^S)$$

F.O.C.:

$$\frac{q}{w_1 - qa_2^S} = \frac{1}{a_2^S}$$

From the BC:

$$a_2^S = \frac{w_1}{2q}$$
$$c_1^S = \frac{w_1}{2}$$
$$c_2^S = \frac{w_1}{2q}$$

Borrower (type B):

$$\max \ln(-qa_2^B) + p(w_2(H) + a_2^B) + (1-p)(w_2(L) + a_2^B)$$

F.O.C.:

$$\frac{q}{qa_2^B} + p + (1-p) = 0$$

First suppose the parameters w_2^H and w_2^L are greater than 1. Then,

$$a_2^B = -1$$
$$c_1^B = q$$
$$c_2^B(H) = w_2^H - 1$$

$$c_2^B(L) = w_2^L - 1$$

Imposing market clearing,

$$\frac{w_1}{2q} - 1 = 0$$

So, the price that clears the market is

$$q = \frac{w_1}{2}$$

Then, the competitive equilibrium allocations are,

$$c_1^S = \frac{w_1}{2}$$

$$c_2^S = 1$$

$$c_1^B = \frac{w_1}{2}$$

$$c_2^B(H) = w_2^H - 1$$

$$c_2^B(L) = w_2^L - 1$$

Now, if the parameters are such that $w_2^L < 1$, then this violates the non-negativity of consumption in the low state. Therefore, the borrower cannot borrow the full amount of $a_2^B = -1$. So suppose $w_2^L < 1$, then $a_2^B = -w_2^L$.

Then,

$$a_2^B = -w_2^L$$

$$c_1^B = q w_2^L$$

$$c_2^B(H) = w_2^H - w_2^L$$

$$c_2^B(L) = 0$$

Imposing market clearing,

$$\frac{w_1}{2q} - w_2^L = 0$$

So, the price that clears the market is

$$q = \frac{w_1}{2w_2^L}$$

Then, the competitive equilibrium allocations are,

$$c_1^S = \frac{w_1}{2}$$

$$c_2^S = w_2^L$$

$$c_1^B = \frac{w_1}{2}$$

$$c_2^B(H) = w_2^H - w_2^L$$

$$c_2^B(L) = 0$$

Note that since $w_2^H > w_2^L$, even if both are less than 1, $a_2^B = -w_2^L$ and the allocations are the same as above.

6. Does the competitive equilibrium implement the efficient allocation? Why or why not?

   Answer:

   Recall, the SPP allocation for $c_2^B(L)$ and $c_2^B(L)$ was any values that satisfy,

$$1 + pc_2^B(H) + (1-p)c_2^B(L) = pw_2^H + (1-p)w_2^L$$

   Substituting CE allocations here gives,

$$1 + p(w_2^H - 1) + (1-p)(w_2^L - 1) = pw_2^H + (1-p)w_2^L$$

   which is true. Therefore, the CE allocation is one of the possible SPP allocations.

   If we're under the case where $w_2^L < 1$, then the SPP allocation calls for

$$c_2^B(H) = c_2^B(L) = 0$$

   where as for CE it is

$$c_2^B(H) = w_2^H - w_2^L$$
$$c_2^B(L) = 0$$

   Therefore, it doesn't implement it.

Case 3 - Bonds without commitment

Now suppose that a borrower can choose not to repay their debt, but if so, she incurs a utility loss K_θ that is state dependent. That is, if the borrower defaults in state $\theta = L$, then the utility cost is K_L. If the borrower defaults in state $\theta = H$, then the utility cost is K_H. Assume that $K_H > 1 > K_L$ and that $w_L > K_L$. The parameterization $K_H > K_L$ is meant to capture that there may be more stigma attached to a wealthy person who defaults.

7. What conditions need to be satisfied for the borrower to choose **not** to default in each of the states $\theta = L$ and $\theta = H$? Can the competitive allocation **with commitment** found in Case 2 be implemented under no commitment? Hint: What are the individual rationality constraints here?

   Answer:

   The utility from not defaulting must in higher than defaulting in each of the states. That is, it must be incentive compatible to not default.

$$\begin{aligned} u(w_2^L + a_2^B) &\geq u(w_2^L) - K_L \\ u(w_2^H + a_2^B) &\geq u(w_2^H) - K_H \end{aligned}$$

   With linear utility these are,

$$\begin{aligned} w_2^L + a_2^B &\geq w_2^L - K_L \\ w_2^H + a_2^B &\geq w_2^H - K_H \end{aligned}$$

With the competitive equilibrium borrowing value $a_2^B = -1$ from Case 2,

$$\begin{aligned} K_L &\geq 1 \\ K_H &\geq 1 \end{aligned}$$

Based on the parametric assumptions on K, the first inequality is violated. Therefore, default would occur in the low state.

The **with commitment** equilibrium from Case 2 has no default, so this means it **cannot** be implemented here when there is no commitment. Specifically, the borrower will not accept $c_2^B(L) = w_2^L - 1$ in the low state and choose $c_2^B(L) = w_2^L$ instead. This also implies markets aren't clearing and the seller's allocation is unattainable in the low state either.

If $w_2^L < 1$, then we just repeat same the logic here with $a_2^B = -w_2^L$. This means the individual rationality constraints will be

$$\begin{aligned} K_L &\geq w_2^L \\ K_H &\geq w_2^L \end{aligned}$$

Since Case 3 question gives parametric assumption which says $w_L > K_L$, there is still default in the low state only.

8. Re-solve for a new competitive equilibrium under which the borrower never defaults. For notation, denote the price of the bond in this new equilibrium with Q. Provide intuition on how this new price Q compares to the price q found in Case 2 under full commitment.

   Answer:

   The additional binding constraint for the borrower is that of the low state. (If this were not binding, limited commitment would play no role and Case 2 equilibrium would follow through.)

   $$w_2^L + a_2^B = w_2^L - K_L$$

   Formally, the borrower's new optimization problem is:

   $$\max \ln(c_1^B) + p(c_2^B(H)) + (1-p)(c_2^B(L))$$
   $$\text{s.t.}$$
   $$c_1^B + Qa_2^B = 0$$
   $$c_2^B(H) = w_2^H + a_2^B$$
   $$c_2^B(L) = w_2^L + a_2^B$$
   $$w_2^L + a_2^B = w_2^L - K_L$$

   However, it follows directly from the binding commitment constraint that,

   $$a_2^B = -K_L$$

   Therefore, from the budget constraints,

   $$c_1^B = K_L Q$$

$$c_2^B(H) = w_2^H - K_L$$
$$c_2^B(L) = w_2^L - K_L$$

The optimization problem for the saver does not change. Therefore, the choices continue to be,

$$a_2^S = \frac{w_1}{2Q}$$
$$c_1^S = \frac{w_1}{2}$$
$$c_2^S = \frac{w_1}{2Q}$$

The asset market clearing condition is,

$$0 = a_2^S + a_2^B = \frac{w_1}{2Q} - K_L$$

Therefore, the new market clearing price is,

$$Q = \frac{w_1}{2K_L}$$

And consumption of the saver in period 2 is,

$$c_2^S = K_L$$

Note that since $K_L < 1$, $q < Q$. Therefore, the interest rate without commitment is lower.

In an equilibrium with no default the agent must borrow less than she would with commitment. Borrowing any more means the saver knows the agent will default in the low state. This means that with less borrowing, there is more competition from savers for relatively less borrowing which makes savers more willing to accept a lower interest rate for their lending.

This last part doesn't change even if $w_2^L < 1$ because the binding constraint still gives us that $a_2^B = -K_L$.

Note: You will see a seminal limited commitment model by Kehoe & Levine with Rishabh in the fourth quarter that is similar to this environment. Here though, we have exogenously assumed that the cost of default is higher in the high state. Therefore, the agent has incentive to default in the low state since it's not too costly then. In contrast, in Kehoe & Levine it is costlier to default in the low state for dynamic reasons, so the limited commitment constraint binds in the high state.