Econ712 - Handout 8

1 Lecture review

1.1 Diamond Dybvig

- Idiosyncratic preference shocks, incomplete markets
- Banks may be able to alleviate market incompleteness

1.2 Kiyotaki Moore

- Limited commitment $\Rightarrow$ Colateral constraints
- Borrowing constraint dependent on prices $\Rightarrow$ Re-allocation has feedback effects through prices

2 Idiosyncratic uncertainty

2.1 Static problem with iid preference shocks

Consider a two period economy with a perfectly storable consumption good. Agents are endowed with w_1 of the good in period 1 and w_2 in period 2. Agents are identical in period 1, and in period 2 are either "happy" with probability π or "sad" with probability $1-\pi$. Period 2 shocks are only realized after agents have made there savings decision and are iid. Happy agents have preferences $\log c_1 + \log(c_2+\epsilon)$, while sad agents have preferences $\log c_1 + \log(c_2-\epsilon)$ for some $\epsilon > 0$.

1. Setup and solve for the planner's allocation

   (a) Planner:

$$\max_{c_1, c_{2s}, c_{2h}, S} \log c_1 + \pi \log(c_{2h}+\epsilon) + (1-\pi)\log(c_{2s}-\epsilon) \quad s.t.$$

$$w_1 = c_1 + S$$
$$w_2 + S = \pi c_{2h} + (1-\pi) c_{2s}$$

Assume conditions on w_1, w_2, π, ϵ s.t. $S > 0$. Consolidate the RC into

$$c_1 + \pi c_{2h} + (1-\pi)c_{2s} = w_1 + w_2 \iff$$
$$c_1 + \pi(c_{2h}+\epsilon) + (1-\pi)(c_{2s}-\epsilon) = w_1 + w_2 + \epsilon(2\pi - 1)$$

From log utility:

$$c_1 = \frac{1}{2}(w_1 + w_2 + \epsilon(2\pi-1))$$
$$\pi(c_{2h}+\epsilon) = \frac{\pi}{2}(w_1 + w_2 + \epsilon(2\pi-1))$$
$$(1-\pi)(c_{2s}-\epsilon) = \frac{1-\pi}{2}(w_1 + w_2 + \epsilon(2\pi-1))$$

$$c_1 = \frac{1}{2}\left(w_1 + w_2 + \epsilon\left(2\pi - 1\right)\right)$$
$$c_{2h} = \frac{1}{2}\left(w_1 + w_2 + \epsilon\left(2\pi - 1\right)\right) - \epsilon$$
$$c_{2s} = \frac{1}{2}\left(w_1 + w_2 + \epsilon\left(2\pi - 1\right)\right) + \epsilon$$

Backing out savings:

$$\begin{aligned} S &= w_1 - c_1 \\ &= \frac{1}{2}\left(w_1 - w_2 - \epsilon\left(2\pi - 1\right)\right) \end{aligned}$$

2. Setup and solve for the household's problem in autarky. Are households better off in autarky compared to the planner's allocation?

   (a) Household:

$$\max_{c_1, c_{2s}, c_{2h}, S} \log c_1 + \pi \log\left(c_{2h} + \epsilon\right) + (1-\pi)\log\left(c_{2s} - \epsilon\right) \quad s.t.$$

$$\begin{aligned} w_1 &= c_1 + S \\ w_2 + S &= c_{2h} \\ w_2 + S &= c_{2s} \end{aligned}$$

   Let $c_{2h} = c_{2s} = c$, so that the objective is $\log c_1 + \pi \log\left(c_2 + \epsilon\right) + (1-\pi)\log\left(c_2 - \epsilon\right)$. Consolidate BC:

$$c_1 + c_2 = w_1 + w_2$$

   FOC:

$$\frac{1}{c_1} = \lambda$$
$$\frac{\pi}{c_2 + \epsilon} + \frac{1-\pi}{c_2 - \epsilon} = \lambda$$
$$\Rightarrow c_1\left[\pi\left(c_2 - \epsilon\right) + (1-\pi)\left(c_2 + \epsilon\right)\right] = \left(c_2 + \epsilon\right)\left(c_2 - \epsilon\right)$$

   Subbing in $c_2 = w_1 + w_2 - c_1$ gives a quadratic in c_1.

   (b) Households are better off under the planner's allocations: The household and planner have the same objective function, and the household's allocation satisfy the resource constraint. So the planner could implement the household allocation, but they choose not to, so the planner's allocations are better by revealed preference.

3. Suppose households can make enforcible contracts among each other in period 1. Setup and solve for the competitive equilibrium

   (a) Assume the presence of an asset that pays out if household is sad.

$$\max_{c_1, c_{2s}, c_{2h}, S} \log c_1 + \pi \log\left(c_{2h} + \epsilon\right) + (1-\pi)\log\left(c_{2s} - \epsilon\right) \quad s.t.$$

$$
\begin{aligned}
w_1 &= c_1 + S + pq \\
w_2 + S &= c_{2h} \\
w_2 + S + q &= c_{2s}
\end{aligned}
$$

Consolidate:

$$
\begin{aligned}
w_2 + c_{2h} - w_2 + q &= c_{2s} \\
c_1 + c_{2h} - w_2 + pq &= w_1
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow c_1 + c_{2h}\left(1-p\right) + c_{2s}p = w_1 + w_2 \\
&\iff c_1 + \left(1-p\right)\left(c_{2h}+\epsilon\right) + p\left(c_{2s}-\epsilon\right) = w_1 + w_2 + \epsilon\left(1-2p\right)
\end{aligned}
$$

Log utility gives

$$
\begin{aligned}
c_1 &= \frac{1}{2}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right) \\
\left(1-p\right)\left(c_{2h}+\epsilon\right) &= \frac{\pi}{2}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right) \\
p\left(c_{2s}-\epsilon\right) &= \frac{1-\pi}{2}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{1}{2}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right) \\
c_{2h} &= \frac{\pi}{2\left(1-p\right)}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right) \\
c_{2s} &= \frac{1-\pi}{2p}\left(w_1 + w_2 + \epsilon\left(1-2p\right)\right) + \epsilon
\end{aligned}
$$

Backing out assets:

$$
\begin{aligned}
S &= c_{2h} - w_2 \\
q &= c_{2s} - w_2 - S \\
&= c_{2s} - c_{2h}
\end{aligned}
$$

(b) Asset market clearing requires

$$
\left(1-\pi\right)q + S = w_1 - c_1
$$

$$
\iff
$$

$$
\left(1-\pi\right)\left(c_{2s}-c_{2h}\right) + c_{2h} - w_2 = w_1 - c_1
$$

$$
\iff
$$

$$
c_1 + \left(1-\pi\right)c_{2s} + \pi c_{2h} = w_1 + w_2
$$

Note that setting $p = 1-\pi$ will give the planner solution, which clears markets. So this $p = 1-\pi$, along with the induced consumption and savings decision, is an equilibrium.

2.2 OG problem with iid endowment shocks

Consider an OG economy with 2 period lived agents. Agents are endowed with w_1 when they are young. When they are old, they are endowed with either w_2 of the consumption good with probability π or 0 with probability $1-\pi$. The endowment shocks are realized when they are old and are iid. Assume $w_1 > w_2$. Agents have preferences $\log c_t^t + \beta \log c_{t+1}^t$. The initial old are endowed with πw_2 of the consumption good and $\bar{M}$ units of valueless but perfectly storable currency. They have preference $\beta \log c_1^0$.

1. Setup and solve for the planner's allocation

   (a) Planner:

$$\max \sum_{t=1}^{\infty} \left\{ \beta \left(\pi \log c_{t,h}^{t-1} + (1-\pi) \log c_{t,l}^{t} \right) + \log c_t^t \right\} \quad s.t.$$

$$\pi c_{t,h}^{t-1} + (1-\pi) c_{t,l}^{t-1} + c_t^t = w_1 + \pi w_2$$

FOC:

$$\lambda_t = \frac{1}{c_t^t}$$

$$\pi \lambda_t = \frac{\beta \pi}{c_{t,h}^{t-1}}$$

$$(1-\pi)\lambda_t = \frac{\beta(1-\pi)}{c_{t,l}^{t-1}}$$

$$c_{t,h}^{t-1} = c_{t,l}^{t-1} = \beta c_t^t$$

$$c_t^t = \frac{1}{1+\beta}(w_1 + \pi w_2)$$

2. Setup and solve for the household's problem

   (a) Household:

$$\max \log c_t^t + \beta \left(\pi \log c_{t+1,h}^t + (1-\pi) \log c_{t+1,l}^t \right) \quad s.t.$$

$$p_t c_t^t + M_{t+1} = p_t w_1$$

$$p_{t+1} c_{t+1,h}^t = p_{t+1} w_2 + M_{t+1}$$

$$p_{t+1} c_{t+1,l}^t = M_{t+1}$$

FOC:

$$\frac{1}{c_t^t} = \lambda_1 p_t$$

$$\frac{\beta \pi}{c_{t+1,h}^t} = p_{t+1} \lambda_2$$

$$\frac{\beta(1-\pi)}{c_{t+1,l}^t} = p_{t+1} \lambda_3$$

$$\lambda_1 = \lambda_2 + \lambda_3$$

$$\Rightarrow \frac{1}{p_t c_t} = \frac{\beta\pi}{p_{t+1}c^t_{t+1,h}} + \frac{\beta(1-\pi)}{p_{t+1}c^t_{t+1,l}}$$

Subbing out for M_{t+1} gives a quadratic in M_{t+1} :

$$(p_{t+1}w_2 + M_{t+1})\, M_{t+1} = \beta\pi\,(p_t w_1 - M_{t+1})\,[\pi M_{t+1} + (1-\pi)\,(p_{t+1}w_2 + M_{t+1})]$$

3. Solve for a steady state competitive equilibrium where the currency is valued

   (a) Market clearing requires $M_{t+1} = \bar{M}$. Steady state requires $c^t_{t+1,l} = M_{t+1}/p_{t+1}$ being constant, so $p_{t+1} = \bar{p}$ for some $\bar{p}$. We need to solve for $\bar{p}$ and then back out consumption allocations. Use

$$\left(\bar{p}w_2 + \bar{M}\right)\bar{M} = \beta\pi\left(\bar{p}w_1 - \bar{M}\right)\left[\pi\bar{M} + (1-\pi)\left(\bar{p}w_2 + \bar{M}\right)\right]$$

   to solve for $\bar{p}$

4. Compare allocations in (1) and (3). Are households better off with the allocations in (3)?

   (a) Households better off in (3). Same reasoning as 2.1 (2).

3 Aggregate uncertainty

3.1 Static problem with iid and aggregate endowment shocks

Consider a two period economy with a perfectly storable consumption good. Agents are endowed with w_1 in period 1 and zw_2 in period 2. Here z, w_2 are random variables that are realized in period 2. w_2 is iid across agents, and takes on value w_h with probability π and w_l with probability $1-\pi$. z is common for all agents, and takes on value z_h with probability γ and z_l with probability $1-\gamma$. Agents have preferences $\log c_1 + \beta \log c_2$. Assume $z_h > z_l, w_h > w_l$, and $w_1 \geq z_h w_h$.

1. Setup and solve for the planner's allocation

   (a) Planner:

$$\max \log c_1 + \beta\left[\gamma\left(\pi \log c_{2hh} + (1-\pi)\log c_{2hl}\right) + (1-\gamma)\left(\pi\log c_{2lh} + (1-\pi)\log c_{2ll}\right)\right] \quad s.t.$$

$$\begin{aligned} c_1 + S &= w_1 \\ \pi c_{2hh} + (1-\pi)c_{2hl} &= z_h\left[\pi w_h + (1-\pi)\,w_l\right] + S \\ \pi c_{2lh} + (1-\pi)c_{2ll} &= z_l\left[\pi w_h + (1-\pi)\,w_l\right] + S \end{aligned}$$

   FOC:

$$\frac{1}{c_1} = \lambda_1$$

$$\frac{\beta\gamma}{c_{2hh}} = \frac{\beta\gamma}{c_{2hl}} = \lambda_2$$
$$\frac{\beta(1-\gamma)}{c_{2lh}} = \frac{\beta(1-\gamma)}{c_{2ll}} = \lambda_3$$

$$\lambda_1 = \lambda_2 + \lambda_3$$

$$\Rightarrow c_{2hh} = c_{2hl} = c_{2h}$$
$$c_{2lh} = c_{2ll} = c_{2l}$$

$$\frac{1}{c_1} = \frac{\beta\gamma}{c_{2h}} + \frac{\beta(1-\gamma)}{c_{2l}}$$

Then

$$\beta c_1 \left(\gamma c_{2l} + (1-\gamma) c_{2h}\right) = c_{2h} c_{2l}$$

With $S = w_1 - c_1$,

$$c_{2h} = z_h \left[\pi w_h + (1-\pi) w_l\right] + w_1 - c_1$$
$$c_{2l} = z_l \left[\pi w_h + (1-\pi) w_l\right] + w_1 - c_1$$

Subbing this in gives a quadratic in c_1.

2. Setup the household's problem in autarky. Without solving, can we say whether households are better off in autarky compared to the planner's allocation?

   (a) Households:

$$\max \log c_1 + \beta \left[\gamma \left(\pi \log c_{2hh} + (1-\pi) \log c_{2hl}\right) + (1-\gamma) \left(\pi \log c_{2lh} + (1-\pi) \log c_{2ll}\right)\right] \quad s.t.$$

$$c_1 + S = w_1$$
$$c_{2hh} = z_h w_h + S$$
$$c_{2hl} = z_h w_l + S$$
$$c_{2lh} = z_l w_h + S$$
$$c_{2ll} = z_l w_l + S$$

   (b) Households are better off under planner's alloc, reasoning by revealed preference from planner's problem

3. Suppose households can make enforcible contracts among each other in period 1. Setup and solve for the competitive equilibrium

   (a) For markets to be complete, we need 4 assets for the 4 states of the household in period 2. 1 asset is the savings tech, so we add another 3. Household BC:

$$c_1 + S + p_{hl} q_{hl} + p_{hh} q_{hh} + p_{lh} q_{lh} = w_1$$
$$c_{2hh} = z_h w_h + S + q_{hh}$$
$$c_{2hl} = z_h w_l + S + q_{hl}$$
$$c_{2lh} = z_l w_h + S + q_{lh}$$
$$c_{2ll} = z_l w_l + S$$

   Solves similar to 2.1 (3).

4 IC/IR constraints

4.1 Collateral constraint

Recall the collateral problem in class: 2 period model; 2 goods: non-storable consumption and storable housing; housing have relative prices q_0 and q_1 in period 0 and 1; households have income 0 and y_1 in period 0 and 1; households can borrow b_1 in period 0 to repay $(1+r)\,b_1$ in period 1. Assume that borrowers cannot commit to repay. But in the event of default, lenders can seize κ times the value of housing $q_1 h_0$.

In period 2, for a given value of b_1 and h_0, when would households default on their debt? Argue how this leads to the collateral constraint

$$b_1 \leq \kappa \frac{q_1 h_0}{1+r}$$

- Household default iff

$$(1+r)\,b_1 > \kappa q_1 h_0 \iff$$
$$b_1 > \kappa \frac{q_1 h_0}{1+r}$$

- Lenders then will not lend b_1 above $\kappa \frac{q_1 h_0}{1+r}$, else household would default. This gives the constraint $b_1 \leq \kappa \frac{q_1 h_0}{1+r}$.

4.2 Some information problems

1. A monopolist can choose the price p and quality q of a good sold. There are two types of consumers, each demanding a single unit of the good: a high type with preference $u_h(q) - p$ and a low type with preference $u_l(q) - p$. The monopolist cannot distinguish between the consumer types, but sets a price and quality schedule $\{(p_l, q_l), (p_h, q_h)\}$ such that high types pick (p_h, q_h) and low types pick (p_l, q_l). What is the IC constraint on the price and quality schedule?

   (a) IC for high:
   $$u_h(q_h) - p_h \geq u_h(q_l) - p_l$$

   (b) IC for low:
   $$u_l(q_l) - p_l \geq u_l(q_h) - p_h$$

2. A government decides on the level of proportional income tax and the level of income transfer to households. Households have preferences $\log c - l$, where c is consumption and l is labour. There are two types of households, a high type that produces $y = A_h l$ and a low type that produces $y = A_l l$. The government cannot distinguish between household types, but can set tax and transfer schedules $\{(\tau_i, T_i)\}$ where τ is the proportional income tax, T is the income transfer, and $i \in \{l, h\}$. Note that the after tax income to households are $A_i(1-\tau_i)l + T_i$. What is the IC constraint on the tax and transfer schedules?

   (a) Denote $c_{i,s}$ and $l_{i,s}$ the optimal consumption and labour decision of type i under schedule s

(b) IC for high:

$$\log c_{hh} - l_{hh} \geq \log c_{hl} - l_{hl}$$

(c) IC for low:

$$\log c_{ll} - l_{ll} \geq \log c_{lh} - l_{lh}$$