Econ712 - Handout 2b

In subsequent weeks, we will be concerned with finding the fixed point of an operator T, where

$$T(v)(x) = \sup_y F(x,y) + \beta v(y) \quad s.t.$$

$$y \in G(x),$$

that is finding v s.t. $T(v)(x) = v(x)$.

1 Contraction mapping theorem

Definition. Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself**.** T is a **contraction mapping** (with modulus β) if $\exists\ \beta \in (0,1)$ s.t. $\rho(Tx, Ty) \le \beta\rho(x,y)\ \forall\ x, y \in S$

Theorem. *If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then T has exactly one fixed point v in S*

Road map:

1. Find fixed point candidate v (construct Cauchy sequence and invoke completeness of (S, ρ))
2. Show that $v = Tv$
3. Show that v is unique

Corollary. *Let (S, ρ) be a complete metric space and $T : S \to S$ is a contraction mapping with fixed point $v \in S$. If $S' \subseteq S$ is closed and $T(S') \subseteq S'$, then $v \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$*

2 Blackwell's sufficiency conditions

Theorem. *Let $X \subseteq R^l$ and let $B(X)$ be a space of bounded functions $f : X \to R$, with the sup norm $||f|| = \max_{x \in X} ||f(x)||$. Let $T : B(X) \to B(X)$ be an operator satisfying*
a. Monotonicity: For any $f, g \in B(X)$ s.t. $f(x) \le g(x)\ \forall x \in X$, we have $T(f)(x) \le T(g)(x)\ \forall x \in X$
b. Discounting: $\exists\, \beta \in (0,1)$ s.t

$$T(f + a)(x) \le T(f)(x) + \beta a \quad \forall f \in B(X), a \ge 0, x \in X$$

Then T is a contraction with modulus β

3 Theorem of the maximum

Definition. A correspondence $\Gamma : X \to Y$ is **lower hemi-continuous (lhc)** at x if $\Gamma(x)$ is nonempty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, $\exists$ a sequence $\{y_n\}$ s.t. $y_n \to y$ and $y_n \in \Gamma(x_n)\ \forall n$

Definition. A compact valued correspondence $\Gamma : X \to Y$ is **upper hemi-continuous (uhc)** at x if $\Gamma(x)$ is nonempty and if, for every every sequence $x_n \to x$ and every sequence $\{y_n\}$ s.t. $y_n \in \Gamma(x_n)\ \forall n$, $\exists$ a convergent subsequence of $\{y_n\}$ whos limit point $y \in \Gamma(x)$

Definition. A correspondence $\Gamma : X \to Y$ is **continuous** at x if it is both uhc and lhc at x

Consider the problem

$$\sup_{y \in \Gamma(x)} f(x, y)$$

Define

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

Theorem. *Let $X \subseteq R^l$ and $Y \subseteq R^m$, let $f : X \times Y \to R$ be a contiuous function, and let $\Gamma : X \to Y$ be a compact valued and continuous correspondence. Then the function $h : X \to R$ defined above is continuous, and the correspondence $G : X \to Y$ defined above is nonempty, compact valued, and uhc*

Roadmap:

1. Show that $G(x)$ is nonempty and compact (Compactness of $\Gamma(x)$ and Weierstrass's extreme value theorem)
2. Show that $G(x)$ is uhc (continuity of $\Gamma(x)$ and $f(x)$)
3. Show that $h(x)$ is continuous ($G(x)$ uhc)

Corollary. *Let $X \subseteq R^l$ and $Y \subseteq R^m$, let $f : X \times Y \to R$ be a contiuous function, and let $\Gamma : X \to Y$ be a compact valued and continuous correspondence. If f is also either strictly quasi-concave or strictly concave, then the correspondence $G : X \to Y$ defined above is single valued (ie a function) and continuous*