ECON 712B: Handout 3

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Here, we rigorously establish the connections between the sequence and recursive formulation of a general dynamic optimization problem. Richard Bellman called these connections the *Principle of Optimality.*

The Principle of Optimality[1]

- Consider a **sequence problem (SP)** that takes the form:

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t. } x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, ...$$
$$x_0 \in X \text{ given.}$$

  - X is the set of possible values for the state variable x.
  - $\Gamma : X \to X$ is the feasible correspondence.
  - $A = \{(x, y) \in X \times X, y \in \Gamma(x)\}$ is the graph of Γ.
  - $F : A \to \mathbb{R}$ is the one-period return function.
  - $\beta \geq 0$ is the stationary discount factor.
  - $\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), t = 0, 1, ...\}$ is the set of plans that are feasible from x_0.

- The corresponding **functional equation (FE)** takes the form:

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)], \forall x \in X$$

- **The Principle of Optimality** is that the solution v to (FE) evaluated at x_0, gives the value of the supremum in (SP) when the initial state in x_0 and that a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ attains the supremum in (SP) if and only if

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}), t = 0, 1, 2, ... \tag{1}$$

Assumption 1. $\Gamma(x)$ *is nonempty, for all* $x \in X$*.*

Assumption 2. *For all* $x_0 \in X$ *and* $\tilde{x} \in \Pi(x_0)$*,* $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ *exists (it may be* $+\infty$ *or* $-\infty$*).*

- Under Assumptions 1 and 2, we can define some notation around the solution to the (SP):

  - For each $n = 0, 1, ...,$ define $u_n : \Pi(x_0) \to \mathbb{R}$ as the partial sum of discounted returns from period 0 through n from feasible plan $\tilde{x}$.

$$u_n(\tilde{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}).$$

  - Define $u : \Pi(x_0) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ as the (infinite) sum of discounted returns from the feasible plan $\tilde{x}$: $u(\tilde{x}) = \lim_{n\to\infty} u_n(\tilde{x})$.
  - Define $v^* : X \to \bar{\mathbb{R}}$ as the supremum in (SP): $v^*(x_0) = \sup_{\tilde{x}\in\Pi(x_0)} u(\tilde{x})$.[2]

[1] This handout draws heavily from section 4.1 of Stokey, Lucas, Prescott. Some simplification here; more details in SLP.
[2] In this handout, we limit our discussion to $v^*(x_0) \in \mathbb{R}$.

- Properties of (unique) v^* solution to (SP):

$$v^*(x_0) \geq u(\tilde{x}), \text{ for all } \tilde{x} \in \Pi(x_0) \tag{2}$$

$$\text{For any } \varepsilon > 0, v^*(x_0) \leq u(\tilde{x}) + \varepsilon, \text{ for some } \tilde{x} \in \Pi(x_0) \tag{3}$$

- Properties of (not necessarily unique) v solution to (FE):

$$v(x_0) \geq F(x_0, y) + \beta v(y), \text{ for all } y \in \Gamma(x_0) \tag{4}$$

$$\text{For any } \varepsilon > 0, v(x_0) \leq F(x_0, y) + \beta v(y) + \varepsilon, \text{ for some } y \in \Gamma(x_0) \tag{5}$$

Lemma 1. *Let $X, \Gamma, F,$ and β satisfy Assumption 2. Then for any $x_0 \in X$ and any $(x_0, x_1, ...) = \tilde{x} \in \Pi(x_0)$,*

$$u(\tilde{x}) = F(x_0, x_1) + \beta u(\tilde{x}')$$

where $\tilde{x}' = (x_1, x_2, ...)$.

- Theorem 1 establishes that the solution to (SP) satisfies the (FE).
- Theorem 2 establishes a partial converse - requires a boundedness condition.
- Theorem 3 establishes that an optimal policy under (SP) also satisfies (1) for $v = v^*$.
- Theorem 4 establishes a partial converse - also requires a boundedness condition.

Theorem 1. *Let $X, \Gamma, F,$ and β satisfy Assumptions 1 and 2. Then the function v^* satisfies (FE).*

Proof strategy: We know v^* satisfies (2) and (3) and we need to show (4) and (5) hold.

Theorem 2. *Let $X, \Gamma, F,$ and β satisfy Assumptions 1 and 2. If v is a solution to (FE) and satisfies*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \forall (x_0, x_1, ...) \in \Pi(x_0), \forall x_0 \in X, \tag{6}$$

then $v = v^*$.

Proof strategy: We know v satisfies (4), (5), and (6) hold and we need to show (2) and (3) hold.

An immediate consequence of Theorem 2 is that the (FE) has at most one solution satisfying (6).

Theorem 3. *Let $X, \Gamma, F,$ and β satisfy Assumptions 1 and 2. Let $\tilde{x}^* \in \Pi(x_0)$ be a feasible plan that attains the supremum in (SP) for initial state x_0. Then*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), t = 0, 1, 2, ... \tag{7}$$

Proof strategy: Establish (7) for $t = 0$ and apply induction to get for all t.

Theorem 4. *Let $X, \Gamma, F,$ and β satisfy Assumptions 1 and 2. Let $\tilde{x}^* \in \pi(x_0)$ be a feasible plan from x_0 satisfying (7) and with*

$$\limsup_{t\to\infty} \beta^t v^*(x_t^*) \leq 0 \tag{8}$$

Then $\tilde{x}^*$ attains the supremum in (SP) for initial state x_0.

Proof strategy: Show that the $v^*(x_0) \leq u(\tilde{x}^*)$ and $v^*(x_0) \geq u(\tilde{x}^*) \implies v^*(x_0) = u(\tilde{x}^*)$